The effects of a sound wave on the compressible boundary layer on a flat plate

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(*Received* 19 *September* 1957)

SUMMARY

The boundary layer on a hot flat plate which is fixed at zero incidence in a slow stream carrying a progressive sound wave is investigated. Formulae are obtained for the skin friction and the heat transfer in the extreme cases when the frequency is very low and very high. In addition, different methods of simplifying the boundary layer equations in unsteady compressible flow are briefly compared.

1. INTRODUCTION

During the last few years there has been a revival of interest in the theory of unsteady laminar boundary-layer flows. The solution for the initial growth of the boundary layer on a body started from rest has long been known, but the recent work has been concerned with the later stages of such a motion and with the boundary layers associated with fluctuating external flows. One of the first contributions to these new developments was made by Moore (1951), who considered the compressible boundary layer on a heat-insulated flat plate moving with variable velocity in a uniform medium. Moore's work has since been supplemented by that of Ostrach (1955) who treats the same problem for an isothermal flat plate. Both these papers are relevant to the boundary layer on a missile whose velocity changes continually during flight.

Another important contribution, dealing with unsteady incompressible boundary layers, has been made by Lighthill (1954). He investigates the boundary layer in plane flow over a fixed cylinder of arbitrary cross-section when the free stream is fluctuating with small amplitude about a steady mean value. For incompressible flow, this problem is mathematically the same as if the cylinder were moving with the same fluctuating velocity in a uniform medium at rest. For this reason, Moore's and Ostrach's results when interpreted for incompressible fluctuating flow link up with Lighthill's predictions concerning a flat plate.

Lighthill mentioned the Rijke tube among the possible fields of application of the theory of fluctuating boundary layers, and it was, in fact, the phenomenon of the Rijke tube that suggested the work described in this paper. Rayleigh (1894) describes how such a tube may be set up. He quotes the case of a tube 5 ft. long and $4\frac{3}{4}$ in. in diameter, open at both ends and held vertically. When a fine wire gauze stretching across the tube about 1 ft. from the bottom was made red hot by a flame, and then the flame was suddenly removed, almost immediately a sound of considerable power was emitted which lasted for several seconds. In the tube there is a stream of air composed of two components, namely the fairly slow convection current up the tube combined with the longitudinal motion associated with standing sound waves. The energy of the sound waves is derived from the varying transfer of heat from the gauze to the surrounding air, and this changes with the fluctuations in the speed of the air current past the gauze. In the Rijke tube, therefore, we have an example of a hot obstacle in a fluctuating air stream. It is true that the Reynolds number for the flow past the fine wires of the gauze described by Rayleigh is below the range of Reynolds numbers for which boundary-layer theory is valid, and therefore the theory which follows really applies to an obstacle of larger longitudinal dimensions than the gauze.

The principal aim of the present paper is to investigate the effect of high wall temperature on the skin friction and heat transfer on a flat plate fixed in a stream carrying sound waves. In addition, we shall take some account of the effects of the fluctuations in density and temperature that accompany the sound waves in the external stream. (In Lighthill's theory, for incompressible flow, the obstacle may be warm, but not hot, and the density and temperature are constant in the external flow.) Our investigation is closely related to, but not equivalent to, Moore's work on a moving plate. For if, in Moore's problem, we change from axes fixed in the medium to axes fixed in the plate, so as to give the appearance of a plate at rest in a moving medium, we thereby introduce an apparent pressure gradient $-\rho\, dU/dt$ in the boundary layer, where ρ is the density and $U(t)$ is the velocity of the plate. On the other hand, the pressure gradient for a fixed plate in a stream $U(t)$ which fluctuates slightly and has a uniform mean density ρ_∞ is $-\rho_\infty\, dU/dt$. The two pressure gradients are different unless $\rho = \rho_\infty$. Thus it is only for incompressible fluids that Moore's and Ostrach's results provide the answers to our present problem.

2. Equations of motion

We shall consider the boundary layer on a cylindrical obstacle fixed in an unsteady stream. An example of such a stream occurs when sound waves are propagated downstream in steady flow past the obstacle, and this particular flow when the obstacle is a flat plate will be investigated in some detail in § 3. The flow is supposed to be perpendicular to the axis of the fixed cylinder, and so only two space coordinates are involved, x measured along the surface of the cylinder from the leading edge, and y measured perpendicular to the surface. The boundary-layer equations then are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p_1}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2}$$

$$\rho\left(\frac{\partial I}{\partial t} + u\frac{\partial I}{\partial x} + v\frac{\partial I}{\partial y}\right) - \left(\frac{\partial p_1}{\partial t} + u\frac{\partial p_1}{\partial x}\right) = \frac{1}{P}\frac{\partial}{\partial y}\left(\mu\frac{\partial I}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

where u and v are the velocity components in the x and y directions, ρ, I and μ are the density, specific enthalpy and viscosity respectively, and P is the Prandtl number, supposed to be constant. The suffix 1 refers to the external stream at the edge of the boundary layer and the external velocity u_1, density ρ_1, pressure p_1 and enthalpy I_1, which are all functions of t and x in general, obey the equations

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial(\rho_1 u_1)}{\partial x} = 0, \tag{4}$$

$$\rho_1\left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x}\right) = -\frac{\partial p_1}{\partial x}, \tag{5}$$

$$\rho_1\left(\frac{\partial I_1}{\partial t} + u_1 \frac{\partial I_1}{\partial x}\right) - \left(\frac{\partial p_1}{\partial t} + u_1 \frac{\partial p_1}{\partial x}\right) = 0. \tag{6}$$

The last equation expresses the fact that the entropy of each fluid elemen in the external stream is conserved, but in the example considered in §3 we shall use the even stronger condition that the external stream is homentropic. To these equations must be added the equation of state

$$p = \frac{\gamma - 1}{\gamma} \rho I, \tag{7}$$

valid for a gas with constant specific heats, and the equation

$$\rho I = \rho_1 I_1, \tag{8}$$

which, in conjunction with (7), expresses the fact that the pressure p does not change across the boundary layer at any station.

It is convenient now to follow Moore's method of analysis. This is an extension of the Howarth (1948) transformation to unsteady boundary layers. First, the y-coordinate is replaced by Y, where

$$Y(t, x, y) = \frac{1}{\rho_\infty} \int_0^y \rho(t, x, s)\, ds \tag{9}$$

in which the suffix ∞ refers to some standard state of the fluid. Later we shall identify this state with the mean state of the free stream. Next, Moore introduces a function ψ satisfying

$$u = \frac{\rho_\infty}{\rho} \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial Y}, \tag{10}$$

which closely corresponds to the stream function in steady plane flow. (For example, mass flux is measured by $\rho_\infty \psi$.) Then, the equation of continuity (1) is satisfied if

$$v = -\frac{\rho_\infty}{\rho}\left(\frac{\partial \psi}{\partial x} + \frac{\partial Y}{\partial t}\right), \tag{11}$$

and, with t, x, Y as the independent variables, (2) and (3) are immediately transformed into

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial Y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial Y}\right)\frac{\partial \psi}{\partial Y} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}\frac{I}{I_1} + \frac{1}{\rho_\infty^2}\frac{\partial}{\partial Y}\left(\mu\rho\frac{\partial^2 \psi}{\partial Y^2}\right), \tag{12}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial Y}\right)I-\frac{1}{\rho_1}\left(\frac{\partial p_1}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial p_1}{\partial x}\right)\frac{I}{I_1}$$
$$=\frac{1}{\rho_\infty^2}\left[\frac{1}{P}\frac{\partial}{\partial Y}\left(\mu\rho\frac{\partial I}{\partial Y}\right)+\mu\rho\left(\frac{\partial^2\psi}{\partial Y^2}\right)^2\right]. \quad (13)$$

These equations may be simplified further by making the special assumption that the viscosity varies as the temperature T across each station in the boundary layer. According to this,

$$\mu=\mu_w T/T_w, \quad (14)$$

where the viscosity at the wall μ_w may be related to the wall temperature T_w by Sutherland's law

$$\frac{\mu_w}{\mu_\infty}=\left(\frac{T_w}{T_\infty}\right)^{3/2}\frac{T_\infty+T_s}{T_w+T_s}, \quad (15)$$

where T_s is a characteristic temperature of the fluid, 114° K for air. The formula (14) which was discussed by Chapman & Rubesin (1949), is a good approximation to the true viscosity–temperature relation (Sutherland's law) near the wall, all along the boundary layer. Of course it is less accurate in the outer part of the boundary layer, where it would be better to use $\mu=\mu_1 T/T_1$, but errors in the viscosity and conductivity there are mitigated by the smallness of the velocity and temperature derivatives. Equations (14) and (15) together imply that

$$\mu=C\mu_\infty I/I_\infty, \quad (16)$$

where

$$C=\chi^{1/2}\frac{T_\infty+T_s}{\chi T_\infty+T_s}, \quad (17)$$

in which $\chi=T_w/T_\infty$. It now follows, with the help of (7) and (8), that

$$\mu\rho=\mu^*\rho_\infty, \quad (18)$$

where

$$\mu^*=C\frac{p_1}{p_\infty}\mu_\infty. \quad (19)$$

When $\mu\rho$ is replaced by $\mu^*\rho_\infty$, according to (18), the boundary layer equations take the simplified form

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial Y}\right)\frac{\partial\psi}{\partial Y}=-\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}\frac{I}{I_1}+C\nu_\infty\frac{p_1}{p_\infty}\frac{\partial^3\psi}{\partial Y^3}, \quad (20)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial Y}\right)I-\frac{1}{\rho_1}\left(\frac{\partial p_1}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial p_1}{\partial x}\right)\frac{I}{I_1}$$
$$=C\nu_\infty\frac{p_1}{p_\infty}\left[\frac{1}{P}\frac{\partial^2 I}{\partial Y^2}+\left(\frac{\partial^2\psi}{\partial Y^2}\right)^2\right]. \quad (21)$$

These are simultaneous differential equations for the mass-flux function ψ and the enthalpy I, which are the two quantities we have to determine.

3. Flat plate fixed in a low speed fluctuating stream

We have already mentioned the flow in the Rijke tube. In order to gain some information about the behaviour of the boundary layer on the obstacle in this kind of flow, we shall examine the boundary layer on a hot flat plate which is fixed at zero incidence in a low speed stream containing a progressive sound wave. We choose a flat plate as the obstacle rather than a circular cylinder, which would be more representative of the wires in the gauze, because the external flow past it is simpler. A flat plate at zero incidence does not, in a first approximation to the boundary-layer problem, affect the incident stream. With a circular cylinder, unless we restricted attention to wavelengths of sound much greater than the radius of the cylinder, we should have to include the effects of the scattering of the sound wave in the velocity external to the boundary layer. (Of course, in the Rijke tube, the sound waves are in fact much longer than the dimensions of the obstacle.) With a flat plate there is no scattering whatever the frequency of the sound waves.

If the stream past the plate has a mean velocity U_∞, and if there is a sound wave of frequency ω moving downstream in it, the resultant velocity may be written, in complex form, as

$$u_1(t,x) = U_\infty\left[1+\epsilon\exp\left(i\omega t-\frac{M}{M+1}is\right)\right], \tag{22}$$

in which $\epsilon \ll 1$. Here, M is the Mach number of the mean flow and s is the frequency parameter $\omega x/U_\infty$. The corresponding pressure and temperature are given by

$$p_1(t,x) = p_\infty\left[1+\gamma M\epsilon\exp\left(i\omega t-\frac{M}{M+1}is\right)\right], \tag{23}$$

$$T_1(t,x) = T_\infty\left[1+(\gamma-1)M\epsilon\exp\left(i\omega t-\frac{M}{M+1}is\right)\right]. \tag{24}$$

These expressions are valid for all values of M and ω; the only approximation that they involve is that ϵ^2 and higher powers of ϵ should be negligible.

We shall confine attention to the case when M is small, partly because this is appropriate to the low speed stream in the Rijke tube and partly because the solution of the boundary layer equations, especially for the temperature, is thereby shortened. In the extreme case $M = 0$, which applies to an incompressible external stream, the speed of sound then being infinite, (22) becomes

$$u_1(t,x) = U_\infty(1+\epsilon e^{i\omega t}), \tag{25}$$

and the corresponding pressure, density, and temperature, are constants with the values p_∞, ρ_∞ and T_∞ respectively. Thus, by setting $M = 0$, the progressive nature of the sound wave is obliterated, and the problem is reduced to that of the compressible boundary layer on a flat plate in an incompressible stream whose velocity is fluctuating in magnitude. This problem for a fluid that is incompressible in the boundary layer as well as in the external stream has been discussed by Lighthill (1954) for a cylindrical obstacle of arbitrary cross-section, and by Gibellato (1955) for a flat plate. By taking $M = 0$ in the following theory we shall be able to show how large temperature differences modify their results.

The simplification to the boundary-layer solution that is possible when M is small is brought about by neglecting the dissipation in the energy equation (21). This is a crucial step that sets a limit to the magnitude of M. For the discarded dissipation is $O(M^2)$, while the terms retained are $O(\chi - 1)$, where $\chi = T_w/T_\infty$, and we shall assume that χ is $O(1)$. For a red-hot wire, χ would be in the region of 2. We are therefore committed to neglecting any term whose weight is $O(M^2)$ compared with those retained.

It follows that the external stream may be expressed as

$$u_1(t, x) = U_\infty(1 + E), \tag{26}$$

$$p_1(t, x) = p_\infty(1 + \gamma ME), \tag{27}$$

$$T_1(t, x) = T_\infty[1 + (\gamma - 1)ME], \tag{28}$$

where $E = \epsilon \exp(i\omega t - Mis)$. This stream, like the one from which it has been derived, is homentropic, and so $\rho_1 dI_1 = dp_1$. Consequently, with dissipation neglected, the energy equation (21) takes the simpler form

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial Y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial Y}\right)\frac{I}{I_1} = \frac{C\nu_\infty}{P}\frac{p_1}{p_\infty}\frac{\partial^2}{\partial Y^2}\left(\frac{I}{I_1}\right). \tag{29}$$

In solving equations (20) and (29) it will be convenient to separate the cases of low and high frequency. The frequency parameter s is equal to kx/M, where k is the wave number. In the Rijke tube the quantity kx, being of the order of the ratio of the diameter of the wire to the length of the tube, is small, but since we are considering M to be also small it is quite possible for s to be large. More precisely, we shall examine the cases of small and large values of s in turn.

Case of low frequency

When s is small, the external stream is given by

$$u_1(t, x) = U_\infty[1 + (1 - Mis)\epsilon e^{i\omega t}], \tag{30}$$

$$p_1(t, x) = p_\infty[1 + \gamma M\epsilon e^{i\omega t}], \tag{31}$$

$$T_1(t, x) = T_\infty[1 + (\gamma - 1)M\epsilon e^{i\omega t}], \tag{32}$$

when terms involving M^2 are neglected. Care is required in evaluating terms like $(-1/\rho_1)\partial p_1/\partial x$ in equation (20). This is equal to

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} = \frac{U_\infty^2}{x}[(1 - M)is - M(is)^2]\epsilon e^{i\omega t}$$

according to (26), and does not vanish as (31) would suggest. The appropriate form of the function ψ in the solution for small s is

$$\psi = (CU_\infty \nu_\infty x)^{1/2}\left[F(\eta) + \epsilon e^{i\omega t}\sum_{n=0}^{\infty}(is)^n f_n(\eta)\right], \tag{33}$$

where $\eta = [U_\infty/(C\nu_\infty x)]^{1/2} Y$. It follows that

$$\frac{\partial \psi}{\partial Y} = u = U_\infty\left[F'(\eta) + \epsilon e^{i\omega t}\sum_{n=0}^{\infty}(is)^n f_n'(\eta)\right], \tag{34}$$

and

$$-\frac{\partial\psi}{\partial x} = \tfrac{1}{2}(CU_\infty \nu_\infty/x)^{1/2} \times$$
$$\times\left[\eta F'(\eta)-F(\eta)+\epsilon e^{i\omega t}\sum_{n=0}^{\infty}(is)^n\{\eta f_n'(\eta)-(2n+1)f_n(\eta)\}\right], \quad (35)$$

where the dashes denote differentiation with respect to η. The temperature may be written in the form

$$T/T_1 = \chi-(\chi-1)G(\eta)+\epsilon e^{i\omega t}\sum_{n=0}^{\infty}(is)^n g_n(\eta). \quad (36)$$

The boundary conditions require that

$$\left.\begin{aligned}
&F(0) = F'(0) = 0, && F'(\infty) = 1;\\
&G(0) = 0, && G(\infty) = 1;\\
&f_0(0) = f_0'(0) = 0, && f_0'(\infty) = 1;\\
&f_1(0) = f_1'(0) = 0, && f_1'(\infty) = -M;\\
&f_n(0) = f_n'(0) = 0, && f_n'(\infty) = 0 \quad (n \geqslant 2);\\
&g_0(0) = -(\gamma-1)M\chi, && g_0(\infty) = 0;\\
&g_n(0) = 0, && g_n(\infty) = 0 \quad (n \geqslant 1).
\end{aligned}\right\} \quad (37)$$

When the expressions (34), (35) and (36) are substituted into (20) we obtain, from the term independent of ϵ, Blasius's equation

$$F''' + \tfrac{1}{2}FF'' = 0. \quad (38)$$

The Blasius function F is tabulated for instance by Schlichting (1955), and the numerical values required in the present problem were taken from that source. Considering next the terms involving ϵ (ϵ^2 is of course neglected), we obtain from the terms independent of s

$$f_0''' + \tfrac{1}{2}Ff_0'' + \tfrac{1}{2}F''f_0 = -\gamma MF''' = \tfrac{1}{2}\gamma MFF'', \quad (39)$$

and, from the terms with the factor s,

$$D_1 f_1 = f_0' - (1-M)[\chi-(\chi-1)G], \quad (40)$$

where D_1 denotes the operator

$$\frac{d^3}{d\eta^3} + \tfrac{1}{2}F\frac{d^2}{d\eta^2} - F'\frac{d}{d\eta} + \tfrac{3}{2}F''.$$

Similarly, equation (29) yields the equations

$$\frac{1}{P}G'' + \tfrac{1}{2}FG' = 0, \quad (41)$$

$$\frac{1}{P}g_0'' + \tfrac{1}{2}Fg_0' = (\chi-1)\left[\tfrac{1}{2}G'f_0 + \frac{\gamma M}{P}G''\right] = \tfrac{1}{2}(\chi-1)G'(f_0-\gamma MF), \quad (42)$$

and

$$D_2\, g_1 = \tfrac{3}{2}(\chi-1)G'f_1 + g_0, \quad (43)$$

where D_2 denotes the operator

$$\frac{1}{P}\frac{d^2}{d\eta^2} + \tfrac{1}{2}F\frac{d}{d\eta} - F'.$$

The required solution of (41) is

$$G = \int_0^\eta [F''(s)]^P\, ds \Big/ \int_0^\infty [F''(s)]^P\, ds, \quad (44)$$

and this function is tabulated in table 1 for $P = 0{\cdot}72$, which has been taken as a representative value of the Prandtl number for air. Now, equations (39) and (42) would still survive if ω were zero, when the external velocity would be $U_\infty(1+\epsilon)$. In fact, f_0 and g_0 represent the perturbations in the steady flow solution when the parameters U_∞ and ν_∞ that occur in the steady solution are changed to $U_\infty(1+\epsilon)$ and $\nu_\infty(1+\gamma M\epsilon)$ respectively. This last expression arises from the fact that the factor $C\nu_\infty p_1/p_\infty$ in (20) and (29), which is $C\nu_\infty$ when the flow is steady, takes the value $C\nu_\infty(1+\gamma M\epsilon)$ in the quasi-steady case ($\omega = 0$). Thus

$$(CU_\infty \nu_\infty x)^{1/2} f_0 = \left[U_\infty \frac{\partial}{\partial U_\infty} + \gamma M \nu_\infty \frac{\partial}{\partial \nu_\infty}\right][(CU_\infty \nu_\infty x)^{1/2} F(\eta)],$$

whence

$$f_0 = \tfrac{1}{2}(F+\eta F') + \tfrac{1}{2}\gamma M(F-\eta F'). \tag{45}$$

Similarly,

$$\begin{aligned} g_0 &= -(\gamma-1)M\chi(1-G) + \left[U_\infty \frac{\partial}{\partial U_\infty} + \gamma M \nu_\infty \frac{\partial}{\partial \nu_\infty}\right][\chi-(\chi-1)G(\eta)] \\ &= -(\gamma-1)M\chi(1-G) - \tfrac{1}{2}(\chi-1)(1-\gamma M)\eta G', \end{aligned} \tag{46}$$

in which the first term arises because g_0 and $\chi-(\chi-1)G$ have to satisfy different boundary conditions.

The solution of (40) may be written as

$$\begin{aligned} f_1 = -f_{11} + \chi f_{12} - M\Big[&\tfrac{1}{4}\gamma\left(\eta - \frac{F'}{F''(0)}\right) + \tfrac{1}{4}(1-\tfrac{1}{4}\gamma)F + \tfrac{3}{4}(1-\tfrac{1}{4}\gamma)\eta F' - \\ &-\tfrac{1}{4}\gamma f_{11} + (\chi - 1 + \tfrac{1}{4}\gamma)f_{12} + (1-\tfrac{1}{4}\gamma)f_{13}\Big], \end{aligned} \tag{47}$$

where f_{11}, f_{12} and f_{13} are the solutions of

$$D_1 f_{11} = G - (\tfrac{1}{2}\eta F'' + F'), \tag{48}$$

$$D_1 f_{12} = G - 1, \tag{49}$$

$$D_1 f_{13} = F'^2 - 1, \tag{50}$$

subject to the boundary conditions $f_{1n}(0) = f'_{1n}(0) = f'_{1n}(\infty) = 0$ for $n = 1, 2, 3$. These three differential equations are solved by first finding the solution of the homogeneous differential equation

$$D_1 f_{10} = 0,$$

subject to $f_{10}(0) = f'_{10}(0) = 0$, $f''_{10}(0) = 1$. Then, to find f_{11} for example, the particular integral $\bar{f}_{11}$, of (48) that satisfies $\bar{f}_{11}(0) = \bar{f}'_{11}(0) = \bar{f}''_{11}(0) = 0$ is next determined, and the required function is given by

$$f_{11} = \bar{f}_{11} - \frac{\bar{f}'_{11}(\infty)}{f'_{10}(\infty)} f_{10}. \tag{51}$$

The functions f_{1n} ($n = 0, 1, 2, 3$) are tabulated in table 1.

Finally, the solution of (43) may be written as

$$g_1 = (\chi-1)(g_{11}-\chi g_{12}) + M[\chi(\gamma-1)g_{13} + (\chi-1)\{\tfrac{3}{4}(1-\tfrac{1}{4}\gamma)\eta G' - \tfrac{1}{4}\gamma g_{11} + \\ + (\chi-1+\tfrac{1}{4}\gamma)g_{12} + \tfrac{3}{2}(1-\tfrac{1}{4}\gamma)g_{14} - \tfrac{3}{8}(\gamma/F''(0))g_{15}\}], \quad (52)$$

where g_{11}, g_{12}, g_{13}, g_{14} and g_{15} are the solutions of

$$D_2\, g_{11} = -\tfrac{1}{2}(3f_{11}+\eta)G', \quad (53)$$

$$D_2\, g_{12} = -\tfrac{3}{2}f_{12}\, G', \quad (54)$$

$$D_2\, g_{13} = G-1, \quad (55)$$

$$D_2\, g_{14} = -f_{13}\, G', \quad (56)$$

$$D_2\, g_{15} = -F'G', \quad (57)$$

subject to the boundary conditions $g_{1n}(0) = g_{1n}(\infty) = 0$ for $n = 1, 2, 3, 4, 5$. The solutions of these are found, as for the corresponding f equations, by first determining the solution of

$$D_2\, g_{10} = 0 \quad (58)$$

subject to $g_{10}(0) = 0$, $g'_{10}(0) = 1$, and then combining it in turn with the particular integrals of (53)–(57) for which $g_{1n}(0) = g'_{1n}(0) = 0$. The functions g_{1n} $(n = 0, 1, ..., 5)$ so determined are tabulated in table 2.

The skin friction is

$$\tau_w(t,x) = \frac{\mu_w \rho_w}{\rho_\infty}\left(\frac{U_\infty}{C\nu_\infty x}\right)^{1/2}\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0} = \rho_\infty U_\infty^2\left(\frac{C\nu_\infty}{U_\infty x}\right)^{1/2}\frac{p_1}{p_\infty}\times \\ \times\left[F''(0) + \epsilon e^{i\omega t}\sum_{n=0}^{\infty}(is)^n f''_n(0)\right], \quad (59)$$

and if $(\tau_w)_s$ denotes the skin friction for steady flow, viz.

$$(\tau_w)_s = 0{\cdot}332\rho_\infty U_\infty^2\left(\frac{C\nu_\infty}{U_\infty x}\right)^{1/2},$$

it follows that

$$\frac{\tau_w}{(\tau_w)_s} = 1 + \epsilon e^{i\omega t}[1{\cdot}5 + (3{\cdot}661\chi - 1{\cdot}105)is + \\ + M\{0{\cdot}7 - (3{\cdot}661\chi + 1{\cdot}228)is\}], \quad (60)$$

for $\gamma = 1{\cdot}4$, provided that s is small enough for s^2 and higher powers to be negligible.

Similarly, the heat transfer from the plate to the air is

$$q_w(t,x) = -\frac{\mu_w \rho_w}{P\rho_\infty}\left(\frac{U_\infty}{C\nu_\infty x}\right)^{1/2}\left(\frac{\partial I}{\partial \eta}\right)_{\eta=0} \\ = \frac{1}{P}\rho_\infty\left(\frac{CU_\infty \nu_\infty}{x}\right)^{1/2}\frac{p_1}{p_\infty}I_1\left[(\chi-1)G'(0) - \epsilon e^{i\omega t}\sum_{n=0}^{\infty}(is)^n g'_n(0)\right], \quad (61)$$

and if $(q_w)_s$ is the heat transfer in steady flow, viz.

$$(q_w)_s = 0{\cdot}296\,\frac{1}{P}\rho_\infty\left(\frac{CU_\infty \nu_\infty}{x}\right)^{1/2}(I_w - I_\infty),$$

η	G	G'	f_{10}	f'_{10}	f''_{10}	f_{11}	f'_{11}
0	0	0·2956	0	0	1	0	0
0·2	0·0591	0·2956	0·020	0·200	1·000	0·007	0·073
0·4	0·1182	0·2953	0·080	0·400	1·000	0·029	0·145
0·6	0·1772	0·2944	0·180	0·600	1·000	0·065	0·213
0·8	0·2359	0·2927	0·320	0·800	1·000	0·114	0·276
1·0	0·2942	0·2898	0·500	1·000	1·001	0·175	0·334
1·2	0·3518	0·2857	0·720	1·200	1·002	0·247	0·383
1·4	0·4084	0·2800	0·980	1·401	1·006	0·328	0·424
1·6	0·4636	0·2726	1·281	1·603	1·013	0·416	0·454
1·8	0·5173	0·2635	1·622	1·807	1·025	0·509	0·474
2·0	0·5689	0·2525	2·003	2·013	1·045	0·605	0·483
2·2	0·6182	0·2398	2·427	2·225	1·077	0·701	0·481
2·4	0·6648	0·2256	2·894	2·445	1·123	0·796	0·468
2·6	0·7083	0·2100	3·406	2·676	1·188	0·888	0·447
2·8	0·7487	0·1933	3·966	2·922	1·274	0·974	0·418
3·0	0·7856	0·1758	4·576	3·187	1·385	1·055	0·383
3·2	0·8190	0·1580	5·242	3·478	1·523	1·127	0·344
3·4	0·8488	0·1403	5·969	3·798	1·688	1·192	0·302
3·6	0·8751	0·1229	6·764	4·155	1·881	1·248	0·261
3·8	0·8980	0·1062	7·634	4·552	2·098	1·296	0·220
4·0	0·9177	0·0906	8·588	4·995	2·338	1·337	0·182
4·2	0·9343	0·0762	9·635	5·489	2·597	1·370	0·148
4·4	0·9482	0·0632	10·787	6·035	2·870	1·396	0·118
4·6	0·9597	0·0517	12·053	6·637	3·154	1·417	0·092
4·8	0·9690	0·0417	13·445	7·297	3·443	1·433	0·070
5·0	0·9765	0·0332	14·976	8·015	3·735	1·445	0·052
5·2	0·9824	0·0260	16·655	8·791	4·026	1·454	0·038
5·4	0·9870	0·0201	18·496	9·625	4·314	1·461	0·028
5·6	0·9905	0·0153	20·509	10·516	4·597	1·465	0·019
5·8	0·9932	0·0115	22·706	11·463	4·875	1·469	0·013
6·0	0·9952	0·0085	25·098	12·466	5·147	1·471	0·009
6·2	0·9966	0·0062	27·696	13·522	5·414	1·472	0·006
6·4	0·9977	0·0045	30·510	14·631	5·676	1·473	0·004
6·6	0·9984	0·0032	33·552	15·792	5·934	1·474	0·002
6·8	0·9990	0·0022	36·830	17·004	6·189	1·474	0·002
7·0	0·9993	0·0015	40·357	18·267	6·440	1·475	0·001
7·2	0·9996	0·0010	44·140	19·580	6·690	1·475	0·000
7·4	0·9997	0·0007	48·192	20·943	6·938	1·475	0·000
7·6	0·9998	0·0004	52·521	22·355	7·185	1·475	0·000
7·8	0·9999	0·0003	57·137	23·817	7·431	1·475	0·000
8·0	1·0000	0·0002	62·051	25·327	7·676	1·475	0·000

Table 1.

η	f''_{11}	f_{12}	f'_{12}	f''_{12}	f_{13}	f'_{13}	f''_{13}
0	0·3671	0	0	1·2159	0	0	1·4599
0·2	0·363	0·023	0·224	1·022	0·028	0·272	1·260
0·4	0·351	0·087	0·410	0·840	0·106	0·504	1·062
0·6	0·331	0·185	0·560	0·669	0·227	0·697	0·868
0·8	0·303	0·309	0·678	0·510	0·382	0·852	0·679
1·0	0·268	0·454	0·765	0·364	0·565	0·969	0·497
1·2	0·226	0·613	0·824	0·229	0·768	1·051	0·325
1·4	0·178	0·782	0·858	0·107	0·984	1·100	0·164
1·6	0·126	0·955	0·868	−0·002	1·206	1·118	0·018
1·8	0·072	1·128	0·858	−0·097	1·429	1·108	−0·113
2·0	0·017	1·297	0·830	−0·179	1·647	1·074	−0·225
2·2	−0·036	1·459	0·787	−0·245	1·857	1·019	−0·318
2·4	−0·085	1·611	0·733	−0·296	2·054	0·948	−0·389
2·6	−0·127	1·751	0·670	−0·331	2·235	0·865	−0·438
2·8	−0·162	1·878	0·601	−0·352	2·399	0·774	−0·466
3·0	−0·187	1·992	0·530	−0·358	2·545	0·680	−0·474
3·2	−0·203	2·090	0·459	−0·352	2·671	0·586	−0·464
3·4	−0·209	2·175	0·390	−0·335	2·779	0·495	−0·440
3·6	−0·207	2·247	0·326	−0·309	2·870	0·411	−0·404
3·8	−0·197	2·306	0·267	−0·278	2·944	0·334	−0·361
4·0	−0·182	2·354	0·214	−0·244	3·004	0·267	−0·314
4·2	−0·162	2·392	0·169	−0·208	3·052	0·208	−0·266
4·4	−0·141	2·422	0·131	−0·174	3·088	0·160	−0·220
4·6	−0·119	2·445	0·100	−0·141	3·116	0·120	−0·177
4·8	−0·098	2·462	0·074	−0·112	3·137	0·089	−0·139
5·0	−0·079	2·475	0·054	−0·087	3·152	0·064	−0·107
5·2	−0·062	2·484	0·039	−0·066	3·163	0·046	−0·080
5·4	−0·047	2·491	0·028	−0·050	3·170	0·032	−0·059
5·6	−0·035	2·496	0·019	−0·036	3·176	0·022	−0·042
5·8	−0·026	2·499	0·013	−0·026	3·179	0·014	−0·030
6·0	−0·018	2·501	0·008	−0·018	3·182	0·010	−0·020
6·2	−0·013	2·502	0·006	−0·012	3·183	0·006	−0·014
6·4	−0·009	2·503	0·003	−0·008	3·184	0·004	−0·009
6·6	−0·006	2·503	0·002	−0·006	3·185	0·002	−0·006
6·8	−0·004	2·504	0·001	−0·004	3·185	0·001	−0·004
7·0	−0·002	2·504	0·001	−0·002	3·186	0·001	−0·002
7·2	−0·002	2·504	0·000	−0·002	3·186	0·000	−0·001
7·4	−0·001	2·504	0·000	−0·001	3·186	0·000	−0·001
7·6	−0·001	2·504	0·000	−0·000	3·186	0·000	−0·000
7·8	−0·000	2·504	0·000	−0·000	3·186	0·000	−0·000
8·0	−0·000	2·504	0·000	−0·000	3·186	0·000	−0·000

Table 1 (*cont.*).

η	g_{10}	g'_{10}	g_{11}	g'_{11}	g_{12}	g'_{12}
0	0	1	0	0·2689	0	0·2484
0·2	0·200	1·000	0·054	0·267	0·050	0·248
0·4	0·400	1·004	0·106	0·260	0·099	0·246
0·6	0·602	1·013	0·157	0·249	0·148	0·239
0·8	0·806	1·031	0·206	0·233	0·194	0·228
1·0	1·015	1·060	0·250	0·213	0·238	0·212
1·2	1·231	1·103	0·290	0·188	0·279	0·189
1·4	1·457	1·162	0·326	0·160	0·314	0·161
1·6	1·697	1·241	0·354	0·128	0·343	0·129
1·8	1·955	1·340	0·376	0·093	0·365	0·092
2·0	2·234	1·462	0·391	0·057	0·380	0·054
2·2	2·541	1·609	0·399	0·021	0·386	0·015
2·4	2·880	1·780	0·400	−0·015	0·386	−0·022
2·6	3·255	1·977	0·393	−0·049	0·378	−0·057
2·8	3·672	2·199	0·380	−0·079	0·363	−0·088
3·0	4·136	2·445	0·362	−0·105	0·343	−0·113
3·2	4·652	2·714	0·339	−0·126	0·318	−0·133
3·4	5·223	3·005	0·312	−0·141	0·290	−0·146
3·6	5·855	3·314	0·283	−0·150	0·261	−0·152
3·8	6·550	3·640	0·252	−0·154	0·230	−0·154
4·0	7·312	3·980	0·222	−0·152	0·200	−0·149
4·2	8·143	4·330	0·192	−0·147	0·170	−0·141
4·4	9·045	4·689	0·163	−0·138	0·143	−0·130
4·6	10·019	5·053	0·137	−0·126	0·118	−0·117
4·8	11·066	5·422	0·113	−0·113	0·096	−0·103
5·0	12·187	5·792	0·092	−0·099	0·077	−0·088
5·2	13·383	6·163	0·073	−0·085	0·061	−0·075
5·4	14·653	6·534	0·058	−0·071	0·047	−0·062
5·6	15·996	6·903	0·045	−0·059	0·036	−0·050
5·8	17·414	7·272	0·034	−0·048	0·027	−0·040
6·0	18·905	7·638	0·026	−0·038	0·020	−0·031
6·2	20·469	8·003	0·019	−0·030	0·015	−0·024
6·4	22·106	8·367	0·014	−0·023	0·010	−0·018
6·6	23·816	8·730	0·010	−0·017	0·007	−0·014
6·8	25·598	9·091	0·007	−0·013	0·005	−0·010
7·0	27·452	9·452	0·005	−0·009	0·003	−0·007
7·2	29·378	9·812	0·003	−0·007	0·002	−0·005
7·4	31·377	10·171	0·002	−0·005	0·001	−0·004
7·6	33·447	10·530	0·001	−0·003	0·001	−0·002
7·8	35·589	10·889	0·000	−0·002	0·000	−0·002
8·0	37·802	11·247	0·000	−0·002	0·000	−0·001

Table 2.

η	g_{13}	g'_{13}	g_{14}	g'_{14}	g_{15}	g'_{15}
0	0	0·7079	0	0·2093	0	0·0956
0·2	0·128	0·568	0·042	0·209	0·019	0·094
0·4	0·228	0·439	0·084	0·207	0·038	0·090
0·6	0·304	0·322	0·124	0·202	0·055	0·084
0·8	0·358	0·217	0·164	0·193	0·071	0·076
1·0	0·392	0·126	0·201	0·179	0·085	0·066
1·2	0·409	0·048	0·235	0·161	0·097	0·055
1·4	0·412	−0·016	0·265	0·137	0·107	0·043
1·6	0·404	−0·067	0·290	0·110	0·114	0·030
1·8	0·386	−0·107	0·309	0·080	0·119	0·018
2·0	0·362	−0·135	0·322	0·047	0·122	0·005
2·2	0·333	−0·153	0·328	0·014	0·122	−0·006
2·4	0·301	−0·163	0·327	−0·018	0·119	−0·017
2·6	0·268	−0·165	0·321	−0·048	0·115	−0·026
2·8	0·235	−0·162	0·309	−0·074	0·109	−0·034
3·0	0·204	−0·154	0·292	−0·096	0·102	−0·040
3·2	0·174	−0·143	0·271	−0·112	0·093	−0·044
3·4	0·147	−0·130	0·247	−0·124	0·084	−0·046
3·6	0·122	−0·116	0·222	−0·130	0·075	−0·047
3·8	0·100	−0·102	0·196	−0·130	0·065	−0·046
4·0	0·081	−0·088	0·170	−0·127	0·056	−0·044
4·2	0·065	−0·074	0·145	−0·120	0·048	−0·041
4·4	0·051	−0·062	0·122	−0·111	0·040	−0·037
4·6	0·040	−0·051	0·101	−0·100	0·033	−0·033
4·8	0·031	−0·041	0·082	−0·088	0·027	−0·029
5·0	0·024	−0·033	0·066	−0·075	0·021	−0·025
5·2	0·018	−0·026	0·052	−0·063	0·017	−0·021
5·4	0·013	−0·020	0·040	−0·052	0·013	−0·017
5·6	0·010	−0·015	0·031	−0·042	0·010	−0·014
5·8	0·007	−0·011	0·023	−0·034	0·008	−0·011
6·0	0·005	−0·008	0·017	−0·026	0·006	−0·008
6·2	0·003	−0·006	0·012	−0·020	0·004	−0·006
6·4	0·002	−0·004	0·009	−0·015	0·003	−0·005
6·6	0·002	−0·003	0·006	−0·011	0·002	−0·004
6·8	0·001	−0·002	0·004	−0·008	0·002	−0·003
7·0	0·001	−0·002	0·003	−0·006	0·001	−0·002
7·2	0·001	−0·001	0·002	−0·004	0·001	−0·001
7·4	0·000	−0·001	0·001	−0·003	0·000	−0·001
7·6	0·000	−0·000	0·001	−0·002	0·000	−0·001
7·8	0·000	−0·000	0·000	−0·002	0·000	−0·000
8·0	0·000	−0·000	0·000	−0·001	0·000	−0·000

Table 2 (*cont*.).

it follows that

$$\frac{q_w}{(q_w)_s} = 1+\epsilon e^{i\omega t}\Big[0{\cdot}5+(0{\cdot}840\chi-0{\cdot}910)is+ + \frac{M}{\chi-1}\{0{\cdot}7\chi-1{\cdot}1-(0{\cdot}840\chi^2-0{\cdot}080\chi+0{\cdot}198)is\}\Big], \quad (62)$$

for $\gamma = 1{\cdot}4$, again provided that s^2 and higher powers are negligible.

Case of high frequency

When the frequency parameter s is large, it is appropriate to expand in inverse powers of s. If $\alpha = (is)^{-1/2}$ and $\beta = (i\omega/C\nu_\infty)^{1/2}Y$, it follows that $\eta = \alpha\beta$, and we may write, instead of (33),

$$\psi = U_\infty\left(\frac{C\nu_\infty}{i\omega}\right)^{1/2}\Big[\frac{1}{\alpha}F(\alpha\beta)+E\sum_{n=0}^{\infty}\alpha^n h_n(\beta)\Big]. \quad (63)$$

Then,

$$\frac{\partial\psi}{\partial Y} = u = U_\infty\Big[F'(\alpha\beta)+E\sum_{n=0}^{\infty}\alpha^n h_n'(\beta)\Big], \quad (64)$$

and

$$-\frac{\partial\psi}{\partial x} = \tfrac{1}{2}(i\omega C\nu_\infty)^{1/2}\Big[\alpha^2\beta F'(\alpha\beta)-\alpha F(\alpha\beta)+E\sum_{n=0}^{\infty}(n\alpha^2+2M)\alpha^n h_n(\beta)\Big]. \quad (65)$$

Corresponding to (36) we write

$$T/T_1 = \chi-(\chi-1)G(\alpha\beta)+E\sum_{n=0}^{\infty}\alpha^n k_n(\beta). \quad (66)$$

For small α,

$$F(\alpha\beta) = \tfrac{1}{2}F''(0)\alpha^2\beta^2+O(\alpha^5),$$

and

$$G(\alpha\beta) = G'(0)\alpha\beta+O(\alpha^4),$$

provided β is not too large. With the help of these approximations, we obtain, by substituting in (20) and considering successive powers of α, the following differential equations

$$h_0'''-h_0' = -\chi(1-M),$$
$$h_0'''-h_1' = (\chi-1)(1-M)G'(0)\beta+MF''(0)(h_0-\beta h_0'),$$
$$h_2'''-h_2' = MF''(0)(h_1-\beta h_1'),$$
$$h_3'''-h_3' = \tfrac{1}{2}F'''(0)(\tfrac{1}{2}\beta^2 h_0''-\beta h_0')+MF''(0)(h_2-\beta h_2').$$

In the complementary function of each differential equation, the term involving e^{β} must be suppressed because it has the wrong behaviour for large β, and therefore there are only two constants of integration. These are determined by the boundary conditions $h_n(0) = h_n'(0) = 0$. The appropriate solutions are

$$h_0 = (1-M)\chi(\beta+e^{-\beta}-1),$$
$$h_1 = -\tfrac{1}{2}(\chi-1)G'(0)\beta^2+M[\tfrac{1}{2}(\chi-1)G'(0)\beta^2+ +\chi F''(0)\{\beta+\tfrac{1}{4}e^{-\beta}(9+5\beta+\beta^2)-\tfrac{9}{4}\}],$$
$$h_2 = -M(\chi-1)F''(0)G'(0)(\beta+\tfrac{1}{6}\beta^3+e^{-\beta}-1),$$
$$h_3 = \tfrac{1}{16}(1-M)\chi F''(0)[4\beta^2+e^{-\beta}(13+13\beta+5\beta^2+\tfrac{2}{3}\beta^3)-13].$$

Similarly by substituting in (29) we obtain the differential equations

$$P^{-1}k_0'' - k_0 = 0,$$

$$P^{-1}k_1'' - k_1 = -M[F''(0)\beta k_0 + (\chi-1)G'(0)h_0],$$

$$P^{-1}k_2'' - k_2 = -M[F''(0)\beta k_1 + (\chi-1)G'(0)h_1],$$

$$P^{-1}k_3'' - k_3 = \tfrac{1}{4}F''(0)\beta^2 k_0' + \tfrac{1}{2}(\chi-1)G'(0)\beta h_0' - M[F''(0)\beta k_2 + (\chi-1)G'(0)h_2].$$

The appropriate solutions, for moderate β, are

$$k_0 = -M(\gamma-1)\chi e^{-\beta\sqrt{P}},$$

$$k_1 = M\chi(\chi-1)G'(0)\left[\beta + \frac{e^{-\beta\sqrt{P}} - Pe^{-\beta}}{1-P} - 1\right],$$

$$k_2 = M(\chi-1)^2[G'(0)]^2\left[\frac{1}{P}(e^{-\beta\sqrt{P}}-1) - \tfrac{1}{2}\beta^2\right],$$

$$k_3 = \tfrac{1}{2}\chi(\chi-1)G'(0)\left[-\beta + \frac{2P}{(1-P)^2}e^{-\beta\sqrt{P}} - \left\{\frac{P}{1-P}\beta + \frac{2P}{(1-P)^2}\right\}e^{-\beta}\right] - M\left[\tfrac{1}{2}\chi(\chi-1)G'(0)\left\{-\beta + \frac{2P}{(1-P)^2}e^{-\beta\sqrt{P}} - \left(\frac{P}{1-P}\beta + \frac{2P}{(1-P)^2}\right)e^{-\beta}\right\} + \tfrac{1}{16}(\gamma-1)\chi F''(0)(\beta + P^{1/2}\beta^2 + \tfrac{2}{3}P\beta^3)e^{-\beta\sqrt{P}}\right],$$

the term $e^{\beta\sqrt{P}}$ in the complementary functions having been discarded in each case, and the solutions having been made to satisfy the boundary conditions

$$k_0(0) = -M(\gamma-1)\chi, \qquad k_n(0) = 0 \quad (n \geqslant 1).$$

It now follows that for large frequencies (α small)

$$\tau_w(t,x) = \frac{\mu_w\rho_w}{\rho_\infty}\left(\frac{i\omega}{C\nu_\infty}\right)^{1/2}\left(\frac{\partial u}{\partial\beta}\right)_{\beta=0} = \rho_\infty U_\infty^2\left(\frac{C\nu_\infty}{U_\infty x}\right)^{1/2}\frac{p_1}{p_\infty}\left[F''(0) + E\sum_{n=0}^{\infty}\alpha^{n-1}h_n''(0)\right],$$

and so

$$\frac{\tau_w}{(\tau_w)_s} = 1 + E[3{\cdot}011\chi(is)^{1/2} - 0{\cdot}8903(\chi-1) + 0{\cdot}3125\chi(is)^{-1} + M\{-3{\cdot}011\chi(is)^{1/2} + 1{\cdot}140\chi + 0{\cdot}5097 - 0{\cdot}2956(\chi-1)(is)^{-1/2} - 0{\cdot}3125\chi(is)^{-1}\}], \quad (67)$$

provided the cube and higher powers of $s^{-1/2}$ are neglected.

Similarly the heat transfer from the plate is

$$q_w(t,x) = -\frac{\mu_w\rho_w}{P\rho_\infty}\left(\frac{i\omega}{C\nu_\infty}\right)^{1/2}\left(\frac{\partial I}{\partial\beta}\right)_{\beta=0} = \frac{1}{P}\rho_\infty\left(\frac{CU_\infty\nu_\infty}{x}\right)^{1/2}\frac{p_1}{p_\infty}I_1\left[(\chi-1)G'(0) - E\sum_{n=0}^{\infty}\alpha^{n-1}k_n'(0)\right],$$

and so

$$\frac{q_w}{(q_w)_s} = 1+E\Big[0\cdot3945\chi(is)^{-1} + \frac{M}{\chi-1}\{-1\cdot148\chi(is)^{1/2}-0\cdot5410\chi^2+ +2\cdot3410\chi-1\cdot8+0\cdot3484(\chi-1)^2(is)^{-1/2}-(0\cdot3945\chi^2- -0\cdot4226\chi)(is)^{-1}\}\Big] \tag{68}$$

for $P = 0\cdot72$, with the cube and higher powers of $s^{-1/2}$ neglected.

If, in order to compare the formulae for small and large frequency, we write

$$\frac{\tau_w}{(\tau_w)_s} = 1+\epsilon e^{i(\omega t-Ms)}(A+iB),$$

and

$$\frac{q_w}{(q_w)_s} = 1+\epsilon e^{i(\omega t-Ms)}(C+iD),$$

where A, B, C and D are real, then

$$\left.\begin{aligned} A &= 1\cdot5+0\cdot7M && \text{(small } s\text{)}\\ &= 2\cdot129s^{1/2}-0\cdot8903(\chi-1)+M[-2\cdot129s^{1/2}+1\cdot1403\chi+ \\ &\qquad +0\cdot5097-0\cdot2090(\chi-1)s^{-1/2}] && \text{(large } s\text{),}\end{aligned}\right\} \tag{69}$$

$$\left.\begin{aligned} B &= (3\cdot661\chi-1\cdot105)s-M(3\cdot661\chi-0\cdot272)s && \text{(small } s\text{)}\\ &= 2\cdot129s^{1/2}-0\cdot3125s^{-1}+M[-2\cdot129s^{1/2}+ \\ &\qquad +0\cdot2090(\chi-1)s^{-1/2}+0\cdot3125s^{-1}] && \text{(large } s\text{),}\end{aligned}\right\} \tag{70}$$

$$\left.\begin{aligned} C &= 0\cdot5+M\frac{0\cdot7\chi-1\cdot1}{\chi-1} && \text{(small } s\text{)}\\ &= M\left[\frac{-0\cdot8117\chi}{\chi-1}s^{1/2}+1\cdot8-0\cdot5410\chi+0\cdot2464(\chi-1)s^{-1/2}\right] && \text{(large } s\text{),}\end{aligned}\right\} \tag{71}$$

$$\left.\begin{aligned} D &= (0\cdot840\chi-0\cdot910)s-\frac{M}{\chi-1}(0\cdot840\chi^2-0\cdot080\chi+0\cdot198)s && \text{(small } s\text{)}\\ &= -0\cdot3945\chi s^{-1}+M\left[\frac{-0\cdot8117\chi}{\chi-1}s^{1/2}-0\cdot2464(\chi-1)s^{-1/2}+ \right.\\ &\qquad \left. +\frac{0\cdot3945\chi^2-0\cdot4225\chi}{\chi-1}s^{-1}\right] && \text{(large } s\text{).}\end{aligned}\right\} \tag{72}$$

The angles $\tan^{-1}(B/A)$ and $\tan^{-1}(D/C)$ are the amounts by which the phases of the skin friction and the heat transfer respectively are in advance of the mainstream fluctuation.

In the important special case $M = 0$,

$$\left.\begin{aligned} \frac{\tau_w}{(\tau_w)_s} &= 1+\epsilon e^{i\omega t}[1\cdot5+i(3\cdot661\chi-1\cdot105)s] && \text{(small } s\text{)}\\ &= 1+\epsilon e^{i\omega t}[2\cdot129s^{1/2}-0\cdot8903(\chi-1)+ \\ &\qquad +i(2\cdot129s^{1/2}-0\cdot3125s^{-1})] && \text{(large } s\text{).}\end{aligned}\right\} \tag{73}$$

When further the flow is completely incompressible, $\chi = 1$, as we have already mentioned, and we then obtain

$$\left.\begin{aligned}\frac{\tau_w}{(\tau_w)_s} &= 1+\epsilon e^{i\omega t}(1{\cdot}5+2{\cdot}556is) && \text{(small } s\text{)}\\ &= 1+\epsilon e^{i\omega t}[2{\cdot}129s^{1/2}+i(2{\cdot}129s^{1/2}-0{\cdot}3125s^{-1})] && \text{(large } s\text{).}\end{aligned}\right\} \quad (74)$$

(We may notice here that Moore (1951) gives 2·555 as the coefficient of is in the formula for small s.) The formulae given by Lighthill for the skin friction in this case are

$$\left.\begin{aligned}\frac{\tau_w}{(\tau_w)_s} &= 1+\epsilon e^{i\omega t}(1{\cdot}5+\tfrac{51}{20}is) && \text{(small } s\text{)}\\ &= 1+\epsilon e^{i\omega t}[2{\cdot}062s^{1/2}+i(2{\cdot}062s^{1/2})] && \text{(large } s\text{).}\end{aligned}\right\} \quad (75)$$

The good agreement between (74) and (75), and particularly the fact that the coefficients of is for low frequencies are practically equal, strongly supports the use of the Pohlhausen approximate method by Lighthill in this problem. Lighthill also showed that the whole range of values of the frequency might reasonably well be covered by using (75) only. For at

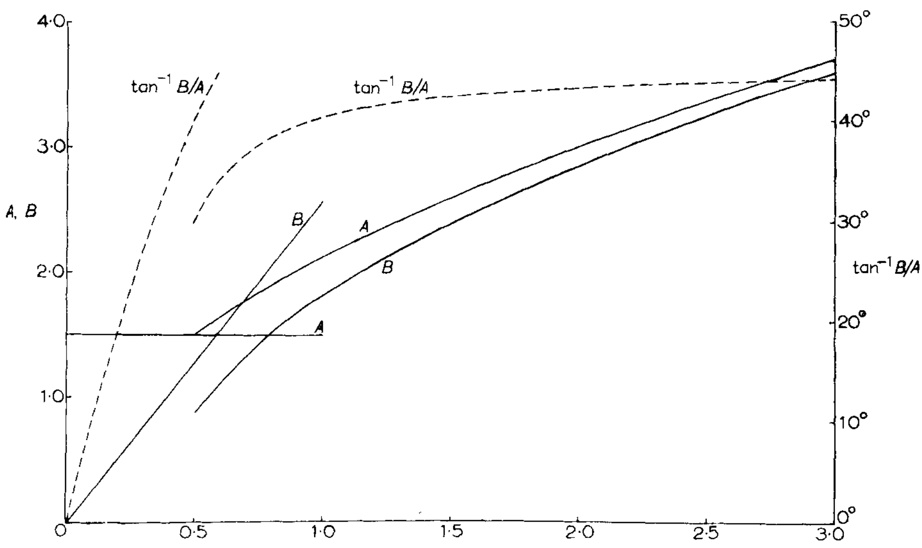

Figure 1. Variation of A, B, and $\tan^{-1}(B/A)$ with frequency parameter s when $M = 0$ and $\chi = 1$.

the place $s = s_0$ where the values of $\tan^{-1}(B/A)$ are equal in the two parts of (75), the values of $(A^2+B^2)^{1/2}$ agree, and so it is reasonable to assume that the two parts of (75) apply to the respective ranges $s < s_0$ and $s > s_0$. This point is illustrated in figure 1, where the quantities A, B and $\tan^{-1}(B/A)$ as given by (74), rather than (75), are plotted for various values of s. For if the term involving s^{-1} in (74) is discarded, so as to bring the formula into line with (75), the B-curve for large s will coincide with the given A-curve for large s. Then, $A = B$ for both large and small s at

$s = s_0 = 0{\cdot}59$ and here the two A-curves are close together. Figure 2, based on (73), shows the same information for $\chi = 2$ as figure 1 does for $\chi = 1$. However, for $\chi = 2$, the whole range of values of s does not appear to be as well represented as it was for $\chi = 1$ simply by the formulae for small and large s. To make a strict comparison with the previous discussion, we should retain only the terms involving $s^{1/2}$ in the part of (73) referring to large s. Then, the A- and B-curves for large s would coincide with the A-curve for large s in figure 1, and we should have $A = B$ for both small and large s at $s = s_0 = 0{\cdot}24$. Here the values of A would be 1·5 for small s and 1·043 for large s (from the A-curve on figure 1). This involves a much larger discrepancy than the case $\chi = 1$, where the corresponding numbers are 1·5 and 1·64 respectively. In fact, figure 2 suggests that the formulae (73) will not be adequate to cover all values of the frequency for an arbitrary value of χ.

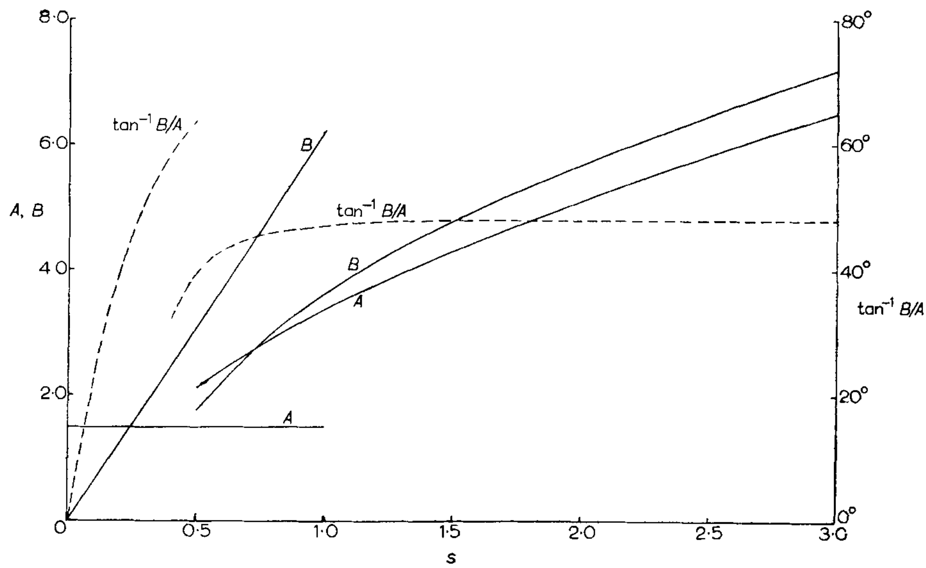

Figure 2. Variation of A, B, and $\tan^{-1}(B/A)$ with frequency parameter s when $M = 0$ and $\chi = 2$.

The quantity $\tan^{-1}(B/A)$ is of course the phase advance of the skin friction relative to the main stream. As (69) and (70) show, this has the asymptotic value $\frac{1}{4}\pi$ for large frequencies whatever the values of χ and M (small). For any specified small value of s, on the other hand, the phase advance is increased by increasing the value of χ, that is, by increasing the temperature of the plate and therefore reducing the inertia of the gas near the surface.

As regards the heat transfer from the wall, it can be seen from (71) and (72) that the phase advance for small frequency is small and negative in the incompressible case ($\chi = 1$) as Lighthill found. In fact, for the case $M = 0$,

$\chi = 1$, Lighthill's work gives

$$\frac{q_w}{(q_w)_s} = 1+\epsilon e^{i\omega t}(0\cdot 5 - 0\cdot 03is) \quad \text{for small } s,$$

whereas the present calculations give the same formula with the coefficient 0·03 replaced by 0·070. (Ostrach's (1955) value is 0·069 to the present order of accuracy.) The Pohlhausen method has in fact predicted this coefficient reasonably well, giving the correct sign and a small quantity of the correct order of magnitude. As χ is increased, the phase advance increases (becoming positive at $\chi = 1\cdot 08$), as it does for the skin friction. Since, for large values of s, C and D are 0 in the case $M = 0$ provided that terms smaller than $s^{1/2}$ are neglected, the heat transfer does not lend itself to a graphical representation corresponding to figures 1 and 2.

Appendix

A note on transformations of the compressible boundary-layer equations for unsteady flow

The analysis of § 2 shows that the Howarth transformation as extended by Moore provides a very useful method of treating the boundary-layer equations for unsteady compressible flow. Now, in the theory of steady boundary layers, besides the Howarth transformation there are the von Mises and the Crocco transformations. All these transformations hinge upon the replacement of the coordinate y perpendicular to the wall by a convenient alternative variable. The Howarth transformation uses $\dfrac{1}{\rho_\infty}\displaystyle\int_0^y \rho\, dy$ instead of y, as we have already seen; the von Mises transformation uses the stream function, which exists in steady plane flow, and the Crocco transformation uses the velocity component u. These have all been described by Howarth (1953). The Crocco transformation has been mainly employed in studying the steady boundary layer on a flat plate, whilst the other two transformations, besides their application to the flat plate, have both been used to show that a compressible boundary layer with a non-zero pressure gradient in certain circumstances may be given in terms of the solution of an associated incompressible boundary-layer flow. It is therefore of some interest to examine whether the von Mises and the Crocco transformations can also be conveniently extended to the case of unsteady boundary-layer flow.

The von Mises equations

In the von Mises transformation for unsteady flow, the new variables are t, x and ψ, where ψ is the mass flux function defined by the equation

$$\frac{\rho u}{\rho_\infty} = \frac{\partial \psi}{\partial y}.$$

In terms of the new variables

$$\rho u = \rho_\infty \left(\frac{\partial y}{\partial \psi}\right)^{-1} \quad \text{and} \quad \rho v = \int_0^\psi \left(\frac{\partial y}{\partial t}\frac{\partial \rho}{\partial \psi} - \frac{\rho_\infty}{\rho u}\frac{\partial \rho}{\partial t}\right) d\psi + \rho u \frac{\partial y}{\partial x}. \tag{76}$$

It is easy to show that the momentum equation (2) becomes

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial \psi}\int_0^{\psi}\frac{1}{u^2}\frac{\partial u}{\partial t}\,d\psi+u\frac{\partial u}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}\frac{I}{I_1}+\frac{u}{\rho_\infty^2}\frac{\partial}{\partial \psi}\left(\mu\rho u\frac{\partial u}{\partial \psi}\right), \tag{77}$$

in which the first two terms on the left-hand side may be replaced by

$$u\frac{\partial}{\partial \psi}\left(u\int_0^{\psi}\frac{1}{u^2}\frac{\partial u}{\partial t}\,d\psi\right),$$

and that the energy equation (3) becomes

$$\frac{\partial I}{\partial t}+u\frac{\partial I}{\partial \psi}\int_0^{\psi}\frac{1}{u^2}\frac{\partial u}{\partial t}\,d\psi+u\frac{\partial I}{\partial x}-\frac{1}{\rho_1}\left(\frac{\partial p_1}{\partial t}+u\frac{\partial p_1}{\partial x}\right)\frac{I}{I_1}$$
$$=\frac{1}{P\rho_\infty^2}u\frac{\partial}{\partial \psi}\left(\mu\rho u\frac{\partial I}{\partial \psi}\right)+\frac{\mu\rho u^2}{\rho_\infty^2}\left(\frac{\partial u}{\partial \psi}\right)^2. \tag{78}$$

Of course these von Mises equations have little practical value if they are in all circumstances more difficult to solve than the corresponding equations (12) and (13) that result from the Howarth transformation. It is scarcely possible to assert that they will be more difficult without a detailed study of some special cases, but it will perhaps be worth while to notice how unsteadiness of the flow increases the difficulties of solution in one or two simple cases of steady flow that have been investigated by means of the von Mises transformation. To take an easy example, consider steady flow past a flat plate for which the momentum equation in the von Mises form is

$$\frac{\partial u}{\partial x}=\frac{1}{\rho_\infty^2}\frac{\partial}{\partial \psi}\left(\mu\rho u\frac{\partial u}{\partial \psi}\right). \tag{79}$$

This equation was solved by von Kármán & Tsien (1938) under the assumptions that $\mu=\mu_\infty(T/T_\infty)^n$ with $n=0{\cdot}76$, $P=1$, and that there is no heat transfer at the plate. The last two conditions ensure that $I+\frac{1}{2}u^2=I_\infty+\frac{1}{2}U_\infty^2$, where the suffix ∞ refers to the uniform main stream, and the first condition shows that $\mu\rho=\mu_\infty\rho_\infty(I/I_\infty)^{n-1}$, so that $\mu\rho$ is a function of u only, given by

$$\mu\rho=\mu_\infty\rho_\infty\left[1+\frac{U_\infty^2}{2I_\infty}\left(1-\frac{u^2}{U_\infty^2}\right)\right]^{n-1}. \tag{80}$$

Thus, (79) reduces to a *second* order differential equation for the single dependent variable u. If the expression (80) is substituted in equation (12), a *third* order equation for ψ results, and so there is something to be said for preferring the von Mises momentum equation in this problem.

In the field of unsteady flow a relatively simple problem of the same type concerns a flat plate in a constant stream when the temperature of the plate is a function of time. Then there are no pressure gradients and the von Mises momentum equation reduces to

$$\frac{\partial}{\partial \psi}\left(u\int_0^{\psi}\frac{1}{u^2}\frac{\partial u}{\partial t}\,d\psi\right)+\frac{\partial u}{\partial x}=\frac{1}{\rho_\infty^2}\frac{\partial}{\partial \psi}\left(\mu\rho u\frac{\partial u}{\partial \psi}\right). \tag{81}$$

Even if we assume that $P=1$, it is no longer possible to write

$$I+\tfrac{1}{2}u^2=I_\infty+\tfrac{1}{2}U_\infty^2$$

now that there is heat transfer at the plate, and so $\mu\rho$ cannot be expressed in terms of u only, as in equation (80).

However, if, instead of assuming that $\mu \propto T^n$, we adopt the approximation referred to in §2 and write

$$\mu\rho = C\mu_\infty \rho_\infty$$

where C, given by (17), is a known function of t only, equation (81) is a differential equation involving only the one dependent variable u. Thus, by altering the assumed viscosity temperature law, we have kept to a problem of the Kármán–Tsien type, namely, the solution of a second-order partial differential equation for u, as far as the momentum equation is concerned. Of course it still remains to solve the energy equation (78), for information about the temperature (and density) in the boundary layer.

Actually, if we use the rather formal viscosity–temperature law $\mu \propto T$, so that $\mu\rho = \mu_\infty \rho_\infty$, this problem can be simplified still further. For the velocity u then loses its explicit dependence on t, and the momentum equation is

$$\frac{\partial u}{\partial x} = \nu_\infty \frac{\partial}{\partial \psi}\left(u \frac{\partial u}{\partial \psi}\right).$$

Thus $u(x, \psi)$ is simply the velocity (expressed in von Mises coordinates) for steady incompressible flow past a flat plate. The temperature would then be given by solving the energy equation

$$\frac{\partial I}{\partial t} + u\frac{\partial I}{\partial x} = \frac{\nu_\infty}{P} u \frac{\partial}{\partial \psi}\left(u \frac{\partial I}{\partial \psi}\right) + \nu_\infty u^2 \left(\frac{\partial u}{\partial \psi}\right)^2.$$

The Crocco equations

In the Crocco transformation the new variables are t, x and z, where z is the non-dimensional velocity distribution function given by

$$z = u(t, x, y)/u_1(t, x). \tag{82}$$

When u_1 is a constant this is precisely the transformation that Crocco used to investigate the steady boundary layer on a flat plate. Following Crocco, we eliminate ρv between the transformed equations of continuity and momentum, and introduce the shearing stress

$$\tau = \mu \frac{\partial u}{\partial y} = \mu u_1 \left(\frac{\partial y}{\partial z}\right)^{-1}$$

as a dependent variable in place of u. In this way, (1) and (2) lead to the equation

$$\frac{\partial u_1}{\partial t} z \frac{\partial}{\partial z}\left(\frac{\mu\rho}{\tau}\right) - u_1 \frac{\partial}{\partial t}\left(\frac{\mu\rho}{\tau}\right) + u_1 \frac{\partial u_1}{\partial x} z^2 \frac{\partial}{\partial z}\left(\frac{\mu\rho}{\tau}\right) - u_1^2 z \frac{\partial}{\partial x}\left(\frac{\mu\rho}{\tau}\right)$$
$$= -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}\frac{\partial}{\partial z}\left(\frac{\mu\rho}{\tau}\frac{I}{I_1}\right) + \frac{1}{u_1}\frac{\partial^2 \tau}{\partial z^2}. \tag{83}$$

Similarly, by transforming (2) and (3) and eliminating ρv between them we obtain

$$\mu\rho\left(u_1\frac{\partial I}{\partial t}-\frac{\partial u_1}{\partial t}z\frac{\partial I}{\partial z}-u_1\frac{\partial u_1}{\partial x}z^2\frac{\partial I}{\partial z}+u_1^2 z\frac{\partial I}{\partial x}\right)-$$
$$-\mu\rho\frac{I}{I_1}\left(\frac{u_1}{\rho_1}\frac{\partial p_1}{\partial t}+\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}\frac{\partial I}{\partial z}+u_1^2\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}z\right)+\frac{1}{u_1}\tau\frac{\partial \tau}{\partial z}\frac{\partial I}{\partial z}$$
$$=\frac{1}{Pu_1}\tau\frac{\partial}{\partial z}\left(\tau\frac{\partial I}{\partial z}\right)+u_1\tau^2. \quad (84)$$

The boundary conditions that go with these equations concern the values of τ and I at the wall, $z=0$, and at the outer edge of the boundary layer $z=1$. Equation (2) shows that $\partial\tau/\partial y=\partial p_1/\partial x$ at the wall, and so

$$\tau\frac{\partial \tau}{\partial z}=\mu_w u_1\frac{\partial p_1}{\partial x} \quad \text{at } z=0. \quad (85)$$

Since the shearing stress vanishes at the outer edge of the boundary layer

$$\tau=0 \quad \text{at } z=1. \quad (86)$$

We shall suppose that either the wall temperature (enthalpy I_w) or the heat flux, q_w, from the wall is specified. It follows that I must satisfy either

$$I=I_w \quad \text{or} \quad (\tau\,\partial I/\partial z)=-Pu_1 q_w \quad \text{at } z=0. \quad (87)$$

Finally, the condition

$$I=I_1 \quad \text{at } z=1 \quad (88)$$

completes the list of four boundary conditions.

Equations (83) and (84) are quite complicated, especially because of the large number of terms on the left-hand side in each case. In this respect they are much worse than either the Howarth–Moore equations (12) and (13) or the von Mises equations (77) and (78). However, it would be feasible to use them in some circumstances. Some preliminary work showed that it would have been practicable to obtain the results of § 3 by solving these equations, but the simpler calculations involved in starting from the Howarth–Moore equations turned out to be preferable. One advantage of the Crocco equations is that the momentum equation is of the second order, but one of the complications encountered in the work just mentioned arose from the frequently occurring term $\mu\rho/\tau$. It will be recalled that in § 3, τ was expressed as a power series, and the reciprocal of such a series, arising in $\mu\rho/\tau$, is another series with complicated coefficients. Of course these difficulties would also be present if the Crocco equations were applied to a steady boundary layer with a non-uniform main stream. So far as the author is aware, even this has not been done.

The Crocco equations are considerably shortened by assuming that the external stream is uniform. In fact for the simple problem, mentioned above, of a flat plate with a varying temperature and with $\mu\rho=\mu_\infty\rho_\infty$, they would reduce to

$$\frac{\partial^2\tau}{\partial z^2}+\mu_\infty\rho_\infty U_\infty^3 z\frac{\partial}{\partial x}\left(\frac{1}{\tau}\right)=0, \quad (89)$$

$$\frac{1}{P}\tau\frac{\partial}{\partial z}\left(\tau\frac{\partial I}{\partial z}\right)+\mu_\infty\rho_\infty U_\infty^3\left(\frac{1}{U_\infty}\frac{\partial I}{\partial t}+z\frac{\partial I}{\partial x}\right)+\tau\frac{\partial \tau}{\partial z}\frac{\partial I}{\partial z}-U_\infty^2\tau^2=0. \quad (90)$$

The solution of (89) for τ is known from the work of Crocco for the case of steady flow, and the problem therefore reduces to solving (90) for I.

In conclusion, the small amount of evidence from the problem considered in this paper suggests that the Howarth–Moore transformation is the simplest to apply of the three available transformations in unsteady compressible boundary-layer theory.

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